

In-domain control of a heat equation: an approach combining zero-dynamics inverse and differential flatness

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Abstract

This paper addresses the set-point control problem of a heat equation with in-domain actuation. The proposed scheme is based on the framework of zero dynamics inverse combined with flat system control. Moreover, the set-point control is cast into a motion planing problem of a multiple-input, multiple-out system, which is solved by a Green's function-based reference trajectory decomposition. The validity of the proposed method is assessed through the convergence and solvability analysis of the control algorithm. The performance of the developed control scheme and the viability of the proposed approach are confirmed by numerical simulation of a representative system.

Keywords: Distributed parameter systems; heat equation; zero-dynamics inverse; differential flatness.

1. Introduction

Control of parabolic partial differential equations (PDEs) is a long-standing problem in PDE control theory and practice. There exists a very rich literature devoted to this topic and it is continuing to draw a great attention for both theoretical studies and practical applications. In the existing literature, the majority of work is dedicated to boundary control, which may be represented as a standard Cauchy problem to which

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functional analytic setting based on semigroup and other related tools can be applied (see, e.g., [1, 2, 3, 4]). It is interesting to note that in recent years, some methods that were originally developed for the control of finite-dimensional systems have been suc-
 10 cessfully extended to the control of parabolic PDEs, such as backstepping (see, e.g., [5, 6, 7]), flat systems (see, e.g., [8, 9, 10, 11, 12, 13, 14]), as well as their variations (see, e.g., [15, 16, 17]).

This paper addresses the in-domain (or interior point) control problem of a heat equation, which may arise in application related concerns for, e.g., the enhancement of
 15 control efficiency. Integrating a number of control inputs acting in the domain will lead to non-standard inhomogeneous PDEs [1, 18], which should be treated differently than the standard boundary control problem. The control scheme developed in the present work is based-on the framework of zero-dynamics inverse (ZDI), which was introduced by Byrnes and his collaborators in [19] and has been exploited and developed in a series
 20 of work [20, 21, 22, 23, 24]. It is pointed out in [23] that “for certain boundary control systems it is very easy to model the system’s zero dynamics, which, in turn, provides a simple systematic methodology for solving certain problems of output regulation.” Indeed, the construction of zero dynamics for output regulation of certain in-domain controlled PDEs is also straightforward (see, e.g., [22]) and hence, the control design
 25 can be carried out in a systematic manner. A main issue related to the ZDI design is that the computation of dynamic control laws requires resolving the corresponding zero dynamics, which may be very difficult for generic regulation problems, such as set-point control. To overcome this difficulty, we leverage one of the fundamental properties of flat systems, that is if a lumped or distributed parameter system is differentially flat (or
 30 flat for short), then its states and inputs can be explicitly expressed by the so-called flat output and its time-derivatives [25, 13]. In the context of ZDI design, the control can be derived from the flat output without explicitly solving the zero dynamics. Moreover, in the framework of flat systems, set-point control can be cast into a problem of motion planning, which can also be carried out in a systematic manner.

35 The system model used in this work is taken from [22]. In order to perform control design, we present the original system in a form of *parallel connection*. This formulation allows a significant simplification of computation. As the control with multiple

actuators located in the domain leads to a multiple-input, multiple-output (MIMO) problem, the design of reference trajectories is not trivial. To overcome this problem, we introduce a Green's function-based reference trajectory decomposition scheme that enables a simple and computational tractable implementation of the proposed control algorithm.

The remainder of the paper is organized as follows. Section 2 describes the model of the considered system and its equivalent settings. Section 3 presents the detailed control design. Section 4 deals with motion planning and addresses the convergence and the solvability of the proposed control scheme. A simulation study is carried out in Section 5, and, finally, some concluding remarks are presented in Section 6.

2. Problem Setting

In the present work, we consider a scalar parabolic equation describing one-dimensional heat transfer with boundary and in-domain control, which is studied in [22]. Denote by $z(x, t)$ the heat distribution over the one-dimensional space, x , and the time, t . The derivatives of $z(x, t)$ with respect to its variables are denoted by z_x and z_t , respectively. For notational simplicity, we may not show all the variables of functions if there is no ambiguity, e.g., $z = z(x, t)$. Consider m points $x_j, j = 1, \dots, m$, in the interval $(0, 1)$ and assume, without loss of generality, that $0 = x_0 < x_1 < x_2 < \dots < x_m < x_{m+1} = 1$. Let $\Omega \doteq \bigcup_{j=0}^m (x_j, x_{j+1})$. The considered heat equation with boundary and in-domain control in a normalized coordinate is of the form:

$$z_t(x, t) - z_{xx}(x, t) = 0, \quad x \in \Omega, \quad t > 0, \quad (1a)$$

$$z(x, 0) = \phi(x), \quad (1b)$$

$$B_0 z = z_x(0, t) - k_0 z(0, t) = 0, \quad (1c)$$

$$B_1 z = z_x(1, t) + k_1 z(1, t) = 0, \quad (1d)$$

$$z(x_j^+) = z(x_j^-), \quad j = 1, \dots, m, \quad (1e)$$

$$B_{x_j} z = [z_x]_{x_j} = u_j(t), \quad j = 1, 2, \dots, m, \quad (1f)$$

where for a function $\psi(\cdot)$ and a point $x \in [0, 1]$ we define

$$[\psi]_x = \psi(x^+) - \psi(x^-),$$

with x^- and x^+ denoting, respectively, the usual meaning of left and right hand limits to x . The initial condition is specified by (1b) with $\phi(x) \in L^2(0, 1)$. In System (1), we assume that we can control the heat flow in and out of the system at the points x_j , i.e.,

$$u_j(t) = [z_x]_{x_j} = z_x(x_j^+, t) - z_x(x_j^-, t).$$

Note that in (1), $B_{x_j}, x_j \in [0, 1]$, represents the point-wise control located on the
50 boundary or in the domain.

The space of weak solutions to System (1) is chosen to be $H^1(0, 1)$. Note that System (1) is exponentially stable in $H^1(0, 1)$ if the boundary controls B_0 and B_1 are chosen such that $k_0 \geq 0$, $k_1 \geq 0$, and $k_0 + k_1 > 0$ [23].

Denote a set of reference signals by $\{z_i^D(x_i, t)\}_{i=1}^m$. Let

$$e_i(t) = z(x_i, t) - z_i^D(x_i, t)$$

be the regulation errors. Let $e(t) = \{e_i(t)\}_{i=1}^m$ and $u(t) = \{u_i(t)\}_{i=1}^m$.

55 **Problem 1.** *The considered regulation problem is to find a dynamic control $u(t)$ such that the regulation error satisfies $e(t) \rightarrow 0$ as $t \rightarrow \infty$.*

The above in-domain control problem can also be formulated in another way by replacing the jump conditions in (1f) by point-wise controls as source terms. The resulting system will be of following the form

$$z_t(x, t) - z_{xx}(x, t) = \sum_{j=1}^m \delta(x - x_j) \alpha_j(t), \quad 0 < x < 1, \quad t > 0, \quad (2a)$$

$$z(x, 0) = \phi(x), \quad (2b)$$

$$B_0 z = z_x(0, t) - k_0 z(0, t) = 0, \quad (2c)$$

$$B_1 z = z_x(1, t) + k_1 z(1, t) = 0, \quad (2d)$$

where $\delta(x - x_j)$ is the Dirac delta function supported at the point x_j , denoting an actuation spot, and $\alpha_j : t \mapsto \mathbb{R}$, $j = 1, \dots, m$, are the in-domain control signals.

Lemma 1. *Considering weak solutions in $H^1(0, 1)$, System (1) and System (2) are*
60 *equivalent if*

$$\alpha_j(t) = -u_j(t) = -[z_x]_{x_j}, \quad j = 1, \dots, m.$$

PROOF. The proof follows the idea presented in [26]. Indeed, it suffices to prove
“System (1) \Rightarrow System (2).” Let $X = L^2(0, 1)$ be a Hilbert space equipped with the
inner product $\langle u, v \rangle = \int_0^1 u(x)v(x)dx$. Let the operator A be defined by $Au = u_{xx}$,
with domain $\mathcal{D}(A) = \{u \in H^2(0, 1); B_0u = B_1u = 0\}$. It is easy to see that A^* , the
65 adjoint of A , is equal to A . Let \tilde{A} be an extension of A with domain $\mathcal{D}(\tilde{A}) = \{u \in$
 $X; u \in H^2(\bigcup_{i=0}^m (x_i, x_{i+1})), B_0u = B_1u = 0, u(x_j^+) = u(x_j^-), j = 1, \dots, m\}$. Let
 $u \in \mathcal{D}(\tilde{A}), v \in \mathcal{D}(A^*) = \mathcal{D}(A)$. Using integration by parts we obtain that

$$\langle \tilde{A}u, v \rangle = \langle u, Av \rangle + \sum_{j=1}^m (u_x(x_j^-) - u_x(x_j^+)) v(x_j). \quad (3)$$

Let $X_{-1} = (\mathcal{D}(A^*))'$, the dual space of $\mathcal{D}(A)$. We need to define another extension
for A . Let $\hat{A} : H^1(0, 1) \rightarrow X_{-1}$ be defined by

$$\langle \hat{A}u, v \rangle = \langle u, A^*v \rangle \quad \text{for all } v \in \mathcal{D}(A^*), \quad (4)$$

with $\mathcal{D}(\hat{A}) = H^1(0, 1)$. Note that $\delta(\cdot - x_j)$ is not in X , but it is in the large space
 X_{-1} . It follows from (3), (4), and $A = A^*$ that

$$\tilde{A}u = \hat{A}u + \sum_{j=1}^m (u_x(x_j^-) - u_x(x_j^+)) \delta(x - x_j), \quad (5)$$

in X_{-1} . If u satisfies System (1), then $\dot{u}(t) = \tilde{A}u(t)$, which yields, considering (5),
 $\dot{u}(t) = \hat{A}u + \sum_{j=1}^m (u_x(x_j^-) - u_x(x_j^+)) \delta(x - x_j)$. Finally, we can see that System (1)
70 becomes System (2) with $\alpha_j(t) = -u_j(t) = -[z_x]_{x_j}, j = 1, \dots, m$, where we look
for generalized solutions $u(x, t) \in \mathcal{D}(\hat{A}) = H^1(0, 1)$ such that (5) is true in X_{-1} . \square

To establish in-domain control at every actuation point, we will proceed in the way

of *parallel connection*, i.e., for every $x_j \in (0, 1)$, consider the following two systems

$$z_t(x, t) - z_{xx}(x, t) = 0, \quad x \in (0, x_j) \cup (x_j, 1), \quad t > 0, \quad (6a)$$

$$z(x, 0) = \phi_j(x), \quad (6b)$$

$$B_0 z = z_x(0, t) - k_0 z(0, t) = 0, \quad (6c)$$

$$B_1 z = z_x(1, t) + k_1 z(1, t) = 0, \quad (6d)$$

$$z(x_j^+) = z(x_j^-), \quad (6e)$$

$$B_{x_j} z = [z_x]_{x_j} = v_j(t). \quad (6f)$$

and

$$z_t(x, t) - z_{xx}(x, t) = \delta(x - x_j) \beta_j(t), \quad 0 < x < 1, \quad t > 0, \quad (7a)$$

$$z(x, 0) = \phi_j(x), \quad (7b)$$

$$B_0 z = z_x(0, t) - k_0 z(0, t) = 0, \quad (7c)$$

$$B_1 z = z_x(1, t) + k_1 z(1, t) = 0, \quad (7d)$$

with $\sum_{j=1}^m \phi_j(x) = \phi(x)$. Similarly, System (6) and (7) are equivalent provided $z \in H^1(0, 1)$ and $\beta_j = -v_j = -[z_x]_{x_j}$. Let $\alpha_j = -u_j = \beta_j = -v_j = -[z_x^j]_{x_j}$ for any $j = 1, 2, \dots, m$, where z^j denotes the solution to System (7). One may directly check that $z(x, t) = \sum_{j=1}^m z^j(x, t)$ is a solution to System (2). Moreover,

$$[z_x]_{x_i} = \sum_{j=1}^m [z_x^j]_{x_i} = [z_x^i]_{x_i} = u_i,$$

for any $i = 1, 2, \dots, m$. Hence $z(x, t) = \sum_{j=1}^m z^j(x, t)$ is a solution to System (1).

Therefore, throughout this paper, we assume $\alpha_j = -u_j = \beta_j = -v_j = -[z_x^j]_{x_j}$ for any $j = 1, 2, \dots, m$. Due to the equivalences of System (1) and (2), and System (6) and

75 (7), we may consider (2) and System (6) in the following parts.

3. Control Design Based on Zero-Dynamics Inverse and Differential Flatness

In the framework of zero-dynamics inverse, the in-domain control is derived from the so-called forced zero-dynamics that are constructed from the original system dynamics by replacing the control constraints by the regulation constraints. To work with

the *parallel connected* system (6), we first split the reference signal as:

$$z^D(x, t) = \sum_{j=1}^m \gamma_j(x, x_j) z_j^d(x_j, t), \quad (8)$$

in which the function $\gamma_j(x, x_j)$ will be determined in Proposition 4 (see Section 4). Denoting by $\varepsilon^j(t) = z^j(x_j, t) - z_j^d(x_j, t)$ the regulation error corresponding to System (6) and replacing the control constraint by $\varepsilon^j(t) = 0$, we obtain the corresponding zero-dynamics for a fixed j :

$$\xi_t(x, t) = \xi_{xx}(x, t), \quad x \in (0, x_j) \cup (x_j, 1), \quad t > 0, \quad (9a)$$

$$\xi(x, 0) = 0, \quad (9b)$$

$$\xi_x(0, t) - k_0 \xi(0, t) = 0, \quad (9c)$$

$$\xi_x(1, t) + k_1 \xi(1, t) = 0, \quad (9d)$$

$$\xi(x_j, t) = z_j^d(x_j, t). \quad (9e)$$

For simplicity, we denote by z^j and ξ^j the solutions to the j^{th} systems (6) and (9), respectively. Also, we write henceforth $z_j^d(t) = z_j^d(x_j, t)$ as the reference trajectory in the j^{th} system (9) if there is no ambiguity. The in-domain control signal of System (6) can then be computed by

$$v_j = [z_x^j]_{x_j} = [\xi_x^j]_{x_j}. \quad (10)$$

Remark 1. Note that for $k_0 \geq 0, k_1 \geq 0$ with $k_0 + k_1 \neq 0$, arguing as [22], we have

$$\varepsilon^j(t) = z^j(x_j, t) - z_j^d(x_j, t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Obviously, to find the control signals, we need to solve the corresponding zero-dynamics (9). For this purpose, we leverage the technique of flat systems [27, 11, 13]. In particular, we apply a standard procedure of Laplace transform-based method to find the solution to (9). Henceforth, we denote by $\hat{f}(x, s)$ the Laplace transform of a function $f(x, t)$ with respect to time variable. Then, for fixed $x_j \in (0, 1)$, the

transformed equations of (9) in the Laplace domain read as

$$s\widehat{\xi}(x, s) = \widehat{\xi}_{xx}(x, s), \quad x \in (0, x_j) \cup (x_j, 1), \quad s \in \mathbb{C}, \quad (11a)$$

$$\widehat{\xi}(x, 0) = 0, \quad (11b)$$

$$\widehat{\xi}_x(0, s) - k_0\widehat{\xi}(0, s) = 0, \quad (11c)$$

$$\widehat{\xi}_x(1, s) + k_1\widehat{\xi}(1, s) = 0, \quad (11d)$$

$$\widehat{\xi}(x_j, s) = \widehat{z}_j^d(x_j, s). \quad (11e)$$

We divide (11) into two sub-systems, i.e., for fixed $x_j \in (0, 1)$, considering

$$s\widehat{\xi}(x, s) = \widehat{\xi}_{xx}(x, s), \quad 0 < x < x_j, \quad s \in \mathbb{C}, \quad (12a)$$

$$\widehat{\xi}(x, 0) = 0, \quad (12b)$$

$$\widehat{\xi}_x(0, s) - k_0\widehat{\xi}(0, s) = 0, \quad (12c)$$

$$\widehat{\xi}(x_j, s) = \widehat{z}_j^d(x_j, s), \quad (12d)$$

and

$$s\widehat{\xi}(x, s) = \widehat{\xi}_{xx}(x, s), \quad x_j < x < 1, \quad s \in \mathbb{C}, \quad (13a)$$

$$\widehat{\xi}(x, 0) = 0, \quad (13b)$$

$$\widehat{\xi}_x(1, s) + k_1\widehat{\xi}(1, s) = 0, \quad (13c)$$

$$\widehat{\xi}(x_j, s) = \widehat{z}_j^d(x_j, s), \quad (13d)$$

Let $\widehat{\xi}_-^j(x, s)$ and $\widehat{\xi}_+^j(x, s)$ be the general solutions to (12) and (13), respectively, and denote their inverse Laplace transforms by $\xi_-^j(x, t)$ and $\xi_+^j(x, t)$. The solution to (9) can be written as

$$\xi^j(x, t) = \xi_-^j(x, t)\chi_{\{(0, x_j)\}} + \xi_+^j(x, t)\chi_{\{(x_j, 1)\}},$$

where

$$\chi(x)_{\{\Omega_j\}} = \begin{cases} 1, & x \in \Omega_j \subseteq (0, 1); \\ 0, & \text{otherwise.} \end{cases}$$

Then at each point $x_i \in (0, 1)$, by (10) and the argument of “*parallel connection*” (see Section 2), we have $[z_x]_{x_i} = \sum_{j=1}^m [z_x^j]_{x_i} = [z_x^i]_{x_i} = [\xi_x^i]_{x_i}$, $i = 1, \dots, m$. Hence the

in-domain control signals of System (1) can be computed by

$$u_i = [z_x]_{x_i} = [\xi_x^i]_{x_i}, \quad i = 1, \dots, m. \quad (14)$$

In the following steps, we present the computation of the solution to System (9), ξ^j . Issues related to the reference trajectory $z^D(x, t)$ for System (1) will be addressed in Section 4.

Note that $\widehat{\xi}_-^j(x, s)$ and $\widehat{\xi}_+^j(x, s)$, the general solutions to (12) and (13), are given by

$$\begin{aligned} \widehat{\xi}_-^j(x, s) &= C_1\phi_1(x, s) + C_2\phi_2(x, s), \\ \widehat{\xi}_+^j(x, s) &= C_3\phi_1(x, s) + C_4\phi_2(x, s), \end{aligned}$$

with

$$\phi_1(x, s) = \frac{\sinh(\sqrt{s}x)}{\sqrt{s}}, \quad \phi_2(x, s) = \cosh(\sqrt{s}x).$$

We obtain by applying (12c) and (12d)

$$C_1\phi_1(x_j, s) + C_2\phi_2(x_j, s) = \widehat{z}_j^d(x_j, s), \quad C_1 - k_0C_2 = 0,$$

which can be written as

$$\begin{pmatrix} \phi_1(x_j, s) & \phi_2(x_j, s) \\ 1 & -k_0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \widehat{z}_j^d(x_j, s) \\ 0 \end{pmatrix}.$$

Let

$$R_-^j = \begin{pmatrix} \phi_1(x_j, s) & \phi_2(x_j, s) \\ 1 & -k_0 \end{pmatrix}$$

and

$$\widehat{z}_j^d(x_j, s) = -\det(R_-^j)\widehat{y}_-^j(x_j, s). \quad (15)$$

We obtain

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{\text{adj}(R_-^j)}{\det(R_-^j)} \begin{pmatrix} \widehat{z}_j^d(x_j, s) \\ 0 \end{pmatrix} = \begin{pmatrix} k_0\widehat{y}_-^j(x_j, s) \\ \widehat{y}_-^j(x_j, s) \end{pmatrix}.$$

Therefore, the solution to (12) can be expressed as

$$\widehat{\xi}_-^j(x, s) = (k_0\phi_1(x) + \phi_2(x))\widehat{y}_-^j(x_j, s). \quad (16)$$

We may proceed in the same way to deal with (13). Indeed, letting

$$R_+^j = \begin{pmatrix} \phi_1(x_j, s) & \phi_2(x_j, s) \\ \phi_2(1, s) + k_1\phi_1(1, s) & s\phi_1(1, s) + k_1\phi_2(1, s) \end{pmatrix}$$

and

$$\widehat{z}_j^d(x_j, s) = \det(R_+^j) \widehat{y}_+^j(x_j, s), \quad (17)$$

85 we get from (13)

$$\begin{pmatrix} C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} (s\phi_1(1, s) + k_1\phi_2(1, s)) \widehat{y}_+^j(x_j, s) \\ -(\phi_2(1, s) + k_1\phi_1(1, s)) \widehat{y}_+^j(x_j, s) \end{pmatrix},$$

and

$$\begin{aligned} \widehat{\xi}_+^j(x, s) &= ((s\phi_1(1, s) + k_1\phi_2(1, s))\phi_1(x) \\ &\quad + (\phi_2(1, s) + k_1\phi_1(1, s))\phi_2(x)) \widehat{y}_+^j(x_j, s). \end{aligned} \quad (18)$$

Applying modulus theory [28, 29] to (15) and (17), we may choose $\widehat{y}_j(x_j, s)$ as the basic output such that

$$\widehat{y}_+^j(x_j, s) = -\det(R_-^j) \widehat{y}_j(x_j, s), \quad (19)$$

$$\widehat{y}_-^j(x_j, s) = \det(R_+^j) \widehat{y}_j(x_j, s). \quad (20)$$

Then, using the property of hyperbolic functions, we obtain from (16) and (18) that

$$\begin{aligned} \widehat{\xi}_-^j(x, s) &= \left(k_1 \frac{\sinh(\sqrt{s}x_j - \sqrt{s})}{\sqrt{s}} - \cosh(\sqrt{s}x_j - \sqrt{s}) \right) \\ &\quad \times \left(k_0 \frac{\sinh(\sqrt{s}x)}{\sqrt{s}} + \cosh(\sqrt{s}x) \right) \widehat{y}_j(x_j, s), \end{aligned} \quad (21)$$

$$\begin{aligned} \widehat{\xi}_+^j(x, s) &= \left(k_1 \frac{\sinh(\sqrt{s}x - \sqrt{s})}{\sqrt{s}} - \cosh(\sqrt{s}x - \sqrt{s}) \right) \\ &\quad \times \left(k_0 \frac{\sinh(\sqrt{s}x_j)}{\sqrt{s}} + \cosh(\sqrt{s}x_j) \right) \widehat{y}_j(x_j, s). \end{aligned} \quad (22)$$

Note that

$$\widehat{\xi}^j(x, s) = \widehat{\xi}_-^j(x, s) \chi_{\{(0, x_j)\}} + \widehat{\xi}_+^j(x, s) \chi_{\{[x_j, 1)\}} \quad (23)$$

is a solution to (11). Using the fact

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!},$$

we obtain

$$\begin{aligned} & \widehat{\xi}^j(x, s) \\ = & \left[\left(k_0 k_1 \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k+1} (x_j - 1)^{2(n-k)+1}}{(2k+1)! [2(n-k)+1]!} s^n - k_0 \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k+1} (x_j - 1)^{2(n-k)}}{(2k+1)! [2(n-k)]!} s^n \right. \right. \\ & + k_1 \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k} (x_j - 1)^{2(n-k)+1}}{(2k)! [2(n-k)+1]!} s^n - \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k} (x_j - 1)^{2(n-k)}}{(2k)! [2(n-k)]!} s^n \Big) \chi_{\{(0, x_j)\}} \\ & + \left(k_0 k_1 \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k+1} (x - 1)^{2(n-k)+1}}{(2k+1)! [2(n-k)+1]!} s^n - k_0 \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k+1} (x - 1)^{2(n-k)}}{(2k+1)! [2(n-k)]!} s^n \right. \\ & \left. \left. + k_1 \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k} (x - 1)^{2(n-k)+1}}{(2k)! [2(n-k)+1]!} s^n - \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k} (x - 1)^{2(n-k)}}{(2k)! [2(n-k)]!} s^n \right) \chi_{\{[x_j, 1]\}} \right] \widehat{y}_j. \end{aligned}$$

It follows that

$$\begin{aligned} & \xi^j(x, t) \\ = & \left[\left(k_0 k_1 \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k+1} (x_j - 1)^{2(n-k)+1}}{(2k+1)! [2(n-k)+1]!} y_j^{(n)} - k_0 \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k+1} (x_j - 1)^{2(n-k)}}{(2k+1)! [2(n-k)]!} y_j^{(n)} \right. \right. \\ & + k_1 \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k} (x_j - 1)^{2(n-k)+1}}{(2k)! [2(n-k)+1]!} y_j^{(n)} - \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k} (x_j - 1)^{2(n-k)}}{(2k)! [2(n-k)]!} y_j^{(n)} \Big) \chi_{\{(0, x_j)\}} \\ & + \left(k_0 k_1 \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k+1} (x - 1)^{2(n-k)+1}}{(2k+1)! [2(n-k)+1]!} y_j^{(n)} - k_0 \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k+1} (x - 1)^{2(n-k)}}{(2k+1)! [2(n-k)]!} y_j^{(n)} \right. \\ & \left. \left. + k_1 \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k} (x - 1)^{2(n-k)+1}}{(2k)! [2(n-k)+1]!} y_j^{(n)} - \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x_j^{2k} (x - 1)^{2(n-k)}}{(2k)! [2(n-k)]!} y_j^{(n)} \right) \chi_{\{[x_j, 1]\}} \right]. \end{aligned} \quad (24)$$

By a direct computation we get

$$\begin{aligned} \left[\widehat{\xi}_x^j \right]_{x_j} &= \left[\left(\frac{k_0 k_1}{\sqrt{s}} + \sqrt{s} \right) \sinh(\sqrt{s}) + (k_0 + k_1) \cosh(\sqrt{s}) \right] \times \widehat{y}_j(x_j, s) \\ &= \left(k_0 k_1 \sum_{n=0}^{\infty} \frac{s^n}{(2n+1)!} + (k_0 + k_1) \sum_{n=0}^{\infty} \frac{s^n}{(2n)!} + \sum_{n=0}^{\infty} \frac{s^{n+1}}{(2n+1)!} \right) \widehat{y}_j(x_j, s). \end{aligned} \quad (25)$$

It follows from (14) that

$$\begin{aligned} u_j(t) &= [\xi_x^j]_{x_j} \\ &= k_0 k_1 \sum_{n=0}^{\infty} \frac{y_j^{(n)}(x_j, t)}{(2n+1)!} + (k_0 + k_1) \sum_{n=0}^{\infty} \frac{y_j^{(n)}(x_j, t)}{(2n)!} + \sum_{n=0}^{\infty} \frac{y_j^{(n+1)}(x_j, t)}{(2n+1)!}. \end{aligned} \quad (26)$$

Remark 2. The reference signal $z_j^d(x_j, t)$ can be derived in the same way from the flat output from (15) and (17). However, as the flatness-based control is driven by flat output, there is no need to explicitly compute $z_j^d(x_j, t)$.

90 4. Motion Planning

For control purpose, we have to choose appreciate reference trajectories, or equivalently the basic outputs. In the present work, we consider the set-point control problem, i.e. to steer the heat distribution to a desired steady-state profile, denoted by $\bar{z}^D(x)$. Without loss of generality, we consider a set of basic outputs of the form:

$$y_j(t) = \bar{y}(x_j) \varphi_j(t), \quad j = 1, \dots, m, \quad (27)$$

where $\varphi_j(t)$ is a smooth function evolving from 0 to 1. Motion planning amounts then to deriving $\bar{y}(x_j)$ from $\bar{z}^D(x)$ and to determining appropriate functions $\varphi_j(t)$, for $j = 1, \dots, m$.

To this aim and due to the equivalence of the systems (1) and (2), we consider the steady-state heat equation corresponding to System (2):

$$\bar{z}_{xx}(x) = \sum_{j=1}^m \delta(x - x_j) \bar{\alpha}_j, \quad 0 < x < 1, \quad t > 0, \quad (28a)$$

$$\bar{z}_x(0) - k_0 \bar{z}(0) = 0, \quad (28b)$$

$$\bar{z}_x(1) + k_1 \bar{z}(1) = 0. \quad (28c)$$

Based on the principle of superposition for linear systems, the solution to the steady-state heat equation (28) can be expressed as:

$$\bar{z}(x) = \int_0^1 \sum_{j=1}^m G(x, \zeta) \delta(\zeta - x_j) \bar{\alpha}_j d\zeta = \sum_{j=1}^m G(x, x_j) \bar{\alpha}_j. \quad (29)$$

where $G(x, \zeta)$ is the Green's function corresponding to (28), which is of the form

$$G(x, \zeta) = \begin{cases} \frac{(k_1\zeta - k_1 - 1)(k_0x + 1)}{k_0 + k_1 + k_0k_1}, & 0 \leq x < \zeta; \\ \frac{(k_1x - k_1 - 1)(k_0\zeta + 1)}{k_0 + k_1 + k_0k_1}, & \zeta \leq x \leq 1. \end{cases} \quad (30)$$

95 Indeed, it is easy to check that $G_{xx}(x, \zeta) = \delta(x - \zeta)$ and $G(x, \zeta)$ satisfies the boundary conditions, $G_x(0, \zeta) - k_0G(0, \zeta) = 0$ and $G_x(1, \zeta) + k_1G(1, \zeta) = 0$, the joint condition, $G(\zeta^+, \zeta) = G(\zeta^-, \zeta)$, and the jump condition, $[G_x(x, \zeta)]_\zeta = 1$.

Taking m distinguished points along the solution to (28), $\bar{z}(x_1, t), \dots, \bar{z}(x_m, t)$, we get

$$\begin{pmatrix} \bar{z}(x_1) \\ \vdots \\ \bar{z}(x_m) \end{pmatrix} = \begin{pmatrix} G(x_1, x_1) & \cdots & G(x_1, x_m) \\ \vdots & \ddots & \vdots \\ G(x_m, x_1) & \cdots & G(x_m, x_m) \end{pmatrix} \begin{pmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_m \end{pmatrix}. \quad (31)$$

Lemma 2. *The matrix $(G(x_i, x_j))_{m \times m}$ chosen as in (31) is invertible. Thus*

$$\begin{pmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_m \end{pmatrix} = \begin{pmatrix} G(x_1, x_1) & \cdots & G(x_1, x_m) \\ \vdots & \ddots & \vdots \\ G(x_m, x_1) & \cdots & G(x_m, x_m) \end{pmatrix}^{-1} \begin{pmatrix} \bar{z}(x_1) \\ \vdots \\ \bar{z}(x_m) \end{pmatrix}. \quad (32)$$

PROOF. For $m = 1$, since $k_0 \geq 0, k_1 \geq 0, k_0 + k_1 > 0$ and $x_1 \in (0, 1)$, it follows that $G(x_1, x_1) = \frac{(k_1x_1 - k_1 - 1)(k_0x_1 + 1)}{k_0 + k_1 + k_0k_1} < 0$. Hence it is invertible. We prove the claim for $m > 1$ by contradiction. Suppose that the matrix $(G(x_i, x_j))_{m \times m}$ is not invertible, then it is of rank $m - 1$ or less. Without loss of generality, we may assume that for some $x_n > x_i$ with $i = 1, \dots, n - 1$, there exist $n - 1$ constants l_1, l_2, \dots, l_{n-1} such that

$$G(x_1, x_n) = \sum_{i=1}^{n-1} l_i G(x_1, x_i), \quad (33a)$$

\vdots

$$G(x_n, x_n) = \sum_{i=1}^{n-1} l_i G(x_n, x_i), \quad (33b)$$

where $1 < n \leq m$ and $\sum_{i=1}^{n-1} l_i^2 > 0$. Let

$$G(x) = G(x, x_n), \quad F(x) = \sum_{i=1}^{n-1} l_i G(x, x_i).$$

(33) shows that $F(x) = G(x)$ at every boundary point of $[x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$.

Note that $F(x)$ is a linear function in $[x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$, and that $G(x) = \frac{(k_1 x_n - k_1 - 1)(k_0 x + 1)}{k_0 + k_1 + k_0 k_1}$ in $[x_1, x_n]$, i.e., $G(x)$ is a linear function in $[x_1, x_n]$. Hence $F(x) \equiv G(x)$ in $[x_1, x_n]$.

By $F(x_1) = G(x_1)$, we get

$$k_1 x_n - k_1 - 1 = \sum_{i=1}^{n-1} l_i (k_1 x_i - k_1 - 1). \quad (34)$$

By $F(x_n) = G(x_n)$, we get

$$k_0 x_n + 1 = \sum_{i=1}^{n-1} l_i (k_0 x_i + 1). \quad (35)$$

Therefore

$$\sum_{i=1}^{n-1} l_i = 1. \quad (36)$$

By $F_x(x_1^+) = G_x(x_1^+)$ and $F_x(x_n^-) = G_x(x_n^-)$, we get

$$\begin{aligned} k_0(k_1 x_n - k_1 - 1) &= k_0 k_1 \sum_{i=1}^{n-1} l_i x_i - k_0(k_1 + 1) \sum_{i=2}^{n-1} l_i + l_1 k_1, \\ &= k_0 k_1 \sum_{i=1}^{n-1} l_i x_i + k_1 \sum_{i=1}^{n-1} l_i. \end{aligned} \quad (37)$$

It follows that $\sum_{i=2}^{n-1} l_i = 0$, which yields, considering (36), $l_1 = 1$. By $F_x(x_2^+) = G_x(x_2) = F_x(x_2^-)$, we deduce

$$\begin{aligned} &l_1 k_1 (k_0 x_1 + 1) + l_2 k_1 (k_0 x_2 + 1) + k_0 \sum_{i=3}^{n-1} l_i (k_1 x_i - k_1 - 1) \\ &= l_1 k_1 (k_0 x_1 + 1) + k_0 \sum_{i=2}^{n-1} l_i (k_1 x_i - k_1 - 1), \end{aligned}$$

which gives $l_2 = 0$. Similarly, by $F_x(x_j^+) = G_x(x_j) = F_x(x_j^-)$ ($j = 3, 4, \dots, n-2$), we obtain $l_3 = l_4 = \dots = l_{n-2} = 0$. Hence $l_{n-1} = 0$. Then we deduce from (37) that

$$k_0(k_1 x_n - k_1 - 1) = k_1(k_0 x_1 + 1). \quad (38)$$

It follows from (35) that

$$k_0 x_1 + 1 = k_0 x_n + 1. \quad (39)$$

We conclude by (38) and (39) that $k_0 + k_1 + k_0 k_1 = 0$, which is a contradiction to
105 $k_0 \geq 0$, $k_1 \geq 0$, and $k_0 + k_1 > 0$. \square

In steady-state, we can obtain from (26) that

$$\bar{u}_j = (k_0 k_1 + k_0 + k_1) \bar{y}(x_j) = -\bar{\alpha}_j. \quad (40)$$

Finally, $\bar{y}(x_j)$ can be computed by (32) and (40) for a given $\bar{z}^D(x)$.

It is worth noting that (32) provides a simple and straightforward way to compute the static control from the prescribed steady-state profile. Indeed, a direct computation
110 can show that applying (32) will result in the same static control obtained in [22] where a *serially connected* model is used.

To ensure the convergence of (24) and (26), we choose the following smooth function as $\varphi_j(t)$:

$$\varphi_j(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{\int_0^t \exp(-1/(\tau(1-\tau)))^\varepsilon d\tau}{\int_0^T \exp(-1/(\tau(1-\tau)))^\varepsilon d\tau}, & \text{if } t \in (0, T) \\ 1, & \text{if } t \geq T \end{cases} \quad (41)$$

which is known as Gevrey function of order $\sigma = 1 + 1/\varepsilon$, $\varepsilon > 0$ (see, e.g., [13]).

Lemma 3. *If the basic outputs $\varphi_j(t)$, $j = 1, \dots, m$, are chosen as Gevrey functions of order $1 < \sigma < 2$, then the infinite series (24) and (26) are convergent.*

PROOF. We prove the convergence of the power series (24) and (26) using Cauchy-Hadamard Theorem. Indeed, it suffices to prove the convergence of

$$\sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{(2k)! [2(n-k)!]} \varphi_j^{(n)}(t). \quad (42)$$

115 Then the convergence of the series in (24) and (26) follows easily using the same argument.

We recall that the bounds of Gevrey functions of order σ are given by [30]

$$\exists K, M > 0, \forall k \in \mathbb{Z}_{\geq 0}, \forall t \in [t_0, T], \left| \varphi^{(k+1)}(t) \right| \leq M \frac{(k!)^\sigma}{K^k}. \quad (43)$$

Denote in (42)

$$b_n = \sum_{k=0}^n \frac{1}{(2k)! [2(n-k)!]} \varphi_j^{(n)}(t).$$

Then, (42) converges if $\limsup_{n \rightarrow \infty} \sqrt[n]{|b_n|} < 1$. Now b_n can be estimated by (43)

$$\begin{aligned} |b_n| &\leq \sum_{k=0}^n \frac{M}{(2k)! [2(n-k)!]} \frac{(n!)^\sigma}{K^n} \\ &\leq M \frac{2^n}{(2n)!} \frac{(n!)^\sigma}{K^n}. \end{aligned}$$

Therefore

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sqrt[n]{|b_n|} &\leq \limsup_{n \rightarrow \infty} \frac{2}{K} M^{1/n} \frac{((n!)^{1/n})^\sigma}{((2n!)^{1/2n})^2} \\ &\leq \limsup_{n \rightarrow \infty} \frac{2}{K} \frac{(n/e)^\sigma}{(2n/e)^2} \\ &= \frac{e^{2-\sigma}}{2K} \limsup_{n \rightarrow \infty} n^{\sigma-2} = \begin{cases} 0, & \sigma < 2, \\ \frac{1}{2K}, & \sigma = 2, \\ \infty, & \sigma > 2, \end{cases} \end{aligned}$$

where in the second inequality we applied Stirling's formula $\sqrt[n]{n!} \simeq (n/e)$. We can conclude by Cauchy-Hadamard Theorem that (42) converges for $\sigma < 2$, and for $\sigma = 2$ if $2K > 1$. The series (42) diverges if $\sigma > 2$. \square

Theorem 4. Assume $k_0 \geq 0$, $k_1 \geq 0$, with $k_0 + k_1 > 0$. Let the basic outputs $\varphi_j(t)$, $j = 1, \dots, m$, be chosen as (41) with an order $1 < \sigma < 2$. Let the reference trajectory of System (1) be given by (8) with

$$\gamma_j(x, x_j) = -\frac{(k_0 k_1 + k_0 + k_1)G(x, x_j)}{(k_0 x_j + 1)(k_0(x_j - 1) - 1)}, \quad j = 1, \dots, m, \quad (44)$$

where $G(x, \zeta)$ is the Green's function defined by (30). Then the regulation error of System (1) with the control given in (26) tends to zero, i.e.,

$$e_i(t) = z(x_i, t) - z_i^D(x_i, t) \rightarrow 0 \text{ as } t \rightarrow \infty,$$

for $i = 1, 2, \dots, m$.

PROOF. By a direct computation we have

$$\begin{aligned}
|e_i(t)| &= |z(x_i, t) - z_i^D(x_i, t)| \\
&= \left| z(x_i, t) + \sum_{j=1}^m \frac{(k_0 k_1 + k_0 + k_1) G(x_i, x_j) z_j^d(x_j, t)}{(k_0 x_j + 1)(k_0(x_j - 1) - 1)} \right| \\
&= \left| z(x_i, t) + \sum_{j=1}^m \frac{(k_0 k_1 + k_0 + k_1) G(x_i, x_j) \xi^j(x_j, t)}{(k_0 x_j + 1)(k_0(x_j - 1) - 1)} \right| \\
&\leq |z(x_i, t) - \bar{z}(x_i)| + \left| \bar{z}(x_i) + (k_0 k_1 + k_0 + k_1) \sum_{j=1}^m G(x_i, x_j) \bar{y}(x_j) \right| \\
&\quad + \left| (k_0 k_1 + k_0 + k_1) \sum_{j=1}^m \frac{G(x_i, x_j) \xi^j(x_j, t)}{(k_0 x_j + 1)(k_0(x_j - 1) - 1)} \right. \\
&\quad \left. - (k_0 k_1 + k_0 + k_1) \sum_{j=1}^m G(x_i, x_j) \bar{y}(x_j) \right|.
\end{aligned}$$

By (40) and (29), it follows

$$\bar{z}(x_i) = -(k_0 k_1 + k_0 + k_1) \sum_{j=1}^m G(x_i, x_j) \bar{y}(x_j).$$

Based on (24), (27), and the property of $\varphi_j(t)$ we have

$$\frac{\xi^j(x_j, t)}{(k_0 x_j + 1)(k_0(x_j - 1) - 1)} \rightarrow \bar{y}(x_j) \text{ as } t \rightarrow \infty.$$

Note that $z(x_i, t) \rightarrow \bar{z}(x_i)$ as $t \rightarrow \infty$. Therefore $|e_i(t)| \rightarrow 0$ as $t \rightarrow \infty$. \square

Remark 3. For any $x \in (0, 1)$, replace x_i by x in the proof of Proposition 4, we can get $|z(x, t) - z^D(x, t)| \rightarrow 0$ as $t \rightarrow \infty$, which shows that the solution $z(x, t)$ of System (1) converges to the reference trajectory $z^D(x, t)$ at every point $x \in (0, 1)$.

125 5. Simulation Study

In the simulation, we implement System (2) with 12 actuators evenly distributed in the domain at the spot points $\{1/13, 2/13, \dots, 12/13\}$. The heat distribution and the time are all represented in normalized coordinates. The numerical implementation is

based on a PDE solver, `pdepe`, in Matlab PDE Toolbox. 200 points in space and 50
 130 points in time are used for the region $[0, 1] \times [0, 2]$ in numerical simulation. The basic
 outputs $\varphi_j(t)$ used in the simulation are Gevrey functions of the same order. In order to
 meet the convergence condition given in Proposition 3, the order of Gevrey functions
 is set to $\sigma = 1.1$. The feedback boundary control gains are chosen as $k_0 = k_1 = 10$.
 The initial condition in simulation is set to $z(x, 0) = \cos(\pi x)$.

135 The desired heat distribution and the evolution of heat distribution of the system
 controlled by the developed algorithm are depicted in Fig. 1. Snapshots of regulation
 errors are presented in Fig. 2, which show that the regulation error tends to 0 along the
 space. The control signals that steer the heat distribute from the initial profile to track
 the desired one are illustrated in Fig. 3. The simulation results show that the system
 140 performs very well with affordable control efforts.

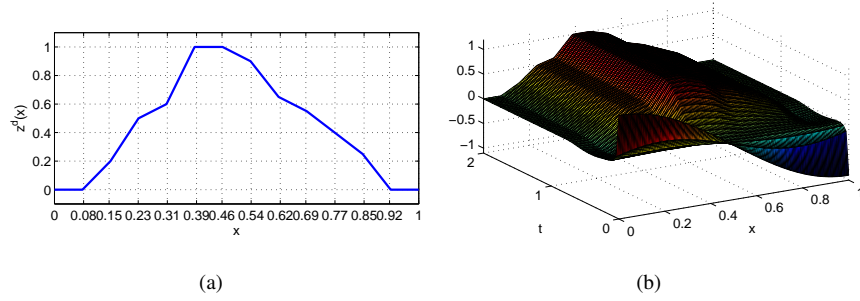


Figure 1: Evolution of heat distribution: (a) desired profile; (b) solution surface.

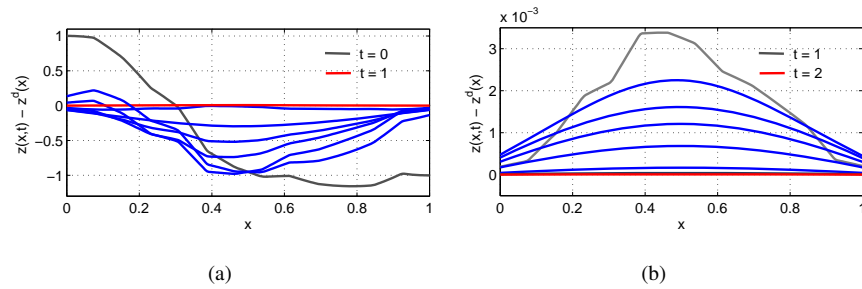


Figure 2: Snapshot of regulation errors: (a) errors for $t \in [0, 1]$; (b) errors for $t \in [1, 2]$.

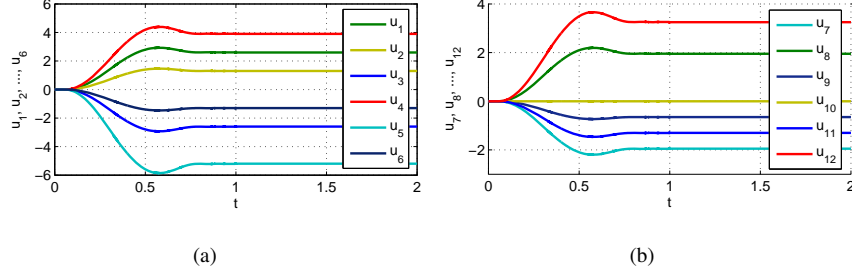


Figure 3: Control signals: (a) u_1 to u_6 ; (b) u_7 to u_{12} .

6. Conclusion

This paper presented a solution to the problem of set-point control of heat distribution with in-domain actuation described by an inhomogeneous parabolic PDE. To apply the paradigm of zero dynamic inverse, the system is presented in an equivalent *parallel* connection form. The technique of flat systems is employed in the design of dynamic control and motion planning. As the control with multiple in-domain actuators results in a MIMO problem, a Green's function-based reference trajectory decomposition is introduced, which considerably simplifies the design and the implementation of the developed control scheme. The convergence and solvability analysis confirms the validity of the control algorithm and the simulation results demonstrate the viability of the proposed approach. Finally, as both ZDI design and flatness-based control can be carried out in a systematic manner, we can expect that the approach developed in this work may be applicable to a broad class of distributed parameter systems.

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